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DECOMPOSITION IN OPTIMAL PLASTIC DESIGN OF STRUCTURES†

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Abstract—A structural synthesis procedure based on the Dantzig-Wolfe decomposition principle is developed for the optimal plastic design of structures subject to multiple load conditions. The decomposed structural synthesis problems consist of a restricted master and a number of subproblems. Each subproblem is further divided into second-level subproblems for which closed form solutions are obtained. The decomposition procedure generates not only an optimal solution for the plastic design problem but also the collapse mechanism associated with the optimal design. An optimal solution generated by the decomposition procedure is shown to be a saddle point for the associated Lagrangian function, which is sufficient for global optimality and zero duality gap. The dual problem for the optimal plastic structural design is interpreted as the maximization of the total power of loads, subject to limitations on the total specific power of dissipation in each structural member. Numerical results for a collection of two- and three-dimensional structures are generated by the decomposition procedure. The computational efficiency and numerical accuracy are confirmed by comparison with previously reported results for trusses and approximate solutions for plane stress structures.

1. INTRODUCTION

Economical use of materials in the construction of aerospace and civil structures has been gaining increased attention during the past two decades. The energy crisis and high material costs in recent years have generated additional incentives for engineers to look for better structural designs. The research work in the field of structural optimization proceeds on two fronts. Some investigators concentrate their efforts on analytical derivation of optimality criteria for simple structures with a few behavior constraints[1–8], others pursue the application of mathematical programming to structural design optimization problems of considerable complexity[8–15]. The analytical treatment provides deeper insight into the nature of optimum structures. The mathematical programming approach leads to interesting results that can be obtained from the structural interpretation of theorems such as duality and complementary slackness[8, 10–12].

A common characteristic shared by most published works on the optimization of large complex structures, based on limit analysis and guarding against plastic collapse, is the theoretical assurance of a global optimal solution. It is the inability to preserve this very desirable property that has prevented previous investigators from going beyond the plastic design of trusses, frames[1, 8–10] and circular plates under axisymmetric conditions[2–5]. Using Von Mises yield criteria, the limit design of membrane plates is a nonlinear non-convex problem. Due to the difficulties in seeking a global optimum, only a small amount of the published work can be found in this area. Brief qualitative treatments of this problem were given in Refs. [8, 9]. The first significant effort in developing a computationally implementable procedure for the optimal plastic design of plane stress structures was reported in Ref. [12]. Duality and solutions were presented for the approximate linear programming (LP) problem which was obtained through piecewise linearization of yield surfaces in the original finite element formulation. However, it was not shown that duality and global optimality of the approximate problem are valid for the original problem. A nested decomposition method for LP problems with staircase constraint matrices was applied in Ref. [13] to the optimal design of

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planar trusses under a single load condition. The nested decomposition method is not applicable for plastic designs under multiple load conditions, since the constraint matrices do not have staircase characteristics.

This paper presents a theoretical analysis and an efficient optimization algorithm for the plastic design of structures which can be represented by truss members, shear panels and plane stress elements. The finite element formulation of limit analysis is based on the lower bound theorem of plasticity theory and elastic-perfectly plastic material behavior. The Von Mises yield criterion is used for plane stress elements because it agrees very well with experimental data for many structural alloys. Global optimal solutions for plastic structural synthesis problems have been efficiently obtained by the application of a formal decomposition method. The optimization algorithm is based on the generalized linear programming approach which is a generalization of the Dantzig–Wolfe decomposition principle. The plastic design problem is decomposed into a restricted master and a number of subproblems. Each subproblem is further decomposed into several second-level subproblems for which closed form solutions have been derived. Due to the sparseness of the constraint matrix, the revised simplex method with product form inverse of the basis matrix and periodic reinversion is used to solve the restricted master problem.

2. FORMULATION OF DESIGN PROBLEM

In the optimal plastic design of structures the structural cost or weight is to be minimized under the constraint that the structures should be able to carry the applied loads without collapse. Since the collapse load of a rigid-perfectly plastic structure coincides with the load-carrying capacity of the corresponding elastic-perfectly plastic structure, the former will be considered in this paper. The collapse load of a structure is the set of applied loads corresponding to a state of impending plastic flow reached in such a way that an increase of plastic strain under constant loads becomes possible for the first time during the loading process. The calculation of collapse load is usually called limit analysis, which is one of the most successful applications of the flow theory of plasticity[16, 17]. Although it is extremely difficult to find exact solutions in the limit analysis of complex structures, the bounds for collapse load can be obtained through the application of the fundamental theorems of limit analysis.

Assume the yield surface is strictly convex so that a unique normal exists at each point on the surface; then the first fundamental theorem provides a lower bound for the exact collapse load. The finite element method has been widely accepted as the most powerful method in the analysis of large complex structures because of its versatility and the relative ease with which complicated geometries and boundary conditions can be handled. The optimal plastic structural design problem, based on a finite element formulation and the lower bound theorem of limit analysis, can be stated as follows for a rather general class of structures:

minimize
$$W = \sum_{j=1}^{J} c_j D_j \tag{1}$$

subject to
$$\sum_{j=1}^{J} \mathbf{Q}_j(\boldsymbol{\sigma}_{ij}) D_j = \mathbf{P}_i \tag{2}$$

$$\boldsymbol{\sigma}_{ij} \in S_{ij} \tag{3}$$

$$D_j \geq 0 \tag{4}$$

$$i = 1, 2, \ldots, I; \quad j = 1, 2, \ldots, J.$$

where

$$\mathbf{Q}_j(\boldsymbol{\sigma}_{ij}) = [B]_j \mathbf{F}_j(\boldsymbol{\sigma}_{ij}) \tag{5}$$

is a vector-valued linear function of the jth element stresses $\boldsymbol{\sigma}_{ij}$ which is the vector of cartesian stresses in the jth element under the ith load condition, $[B]_j$ is a boolean assembly matrix, $\mathbf{P}_i$ is an $M \times 1$ load vector for the ith load condition and D_j represents the jth design variable. The explicit expressions for the unit element force vectors $\mathbf{F}_j(\sigma_{ij})$ are given in Ref. [18] for a truss member, a rectangular shear panel and a constant stress triangular (CST) element. The equilibrium in the finite element model is enforced by eqn (2). The objective function W represents the total structural weight.

The sets S_{ij} are compact convex sets defined by yield criteria

$$S_{ij} = \{\sigma_{ij} | \Phi_j(\sigma_{ij}) \le \bar{\sigma}_j^2\}. \tag{6}$$

The yield stress $\bar{\sigma}_j$ is usually obtained in uniaxial tension tests. The yield functions Φ_j based on the Von Mises yield criterion are positive definite functions of σ_{ij}. Therefore, the yield surface is strictly convex.

It should be pointed out that all the requirements in the first fundamental theorem of limit analysis are satisfied by the finite element structural models made of truss members or shear panels. However, the stress continuity and equilibrium are not exactly satisfied by finite element models involving CST elements or more than one type of element. The stress field is continuous and in equilibrium within each element, but not across the element interfaces. The overall equilibrium is approximately represented by eqns (2) at the nodal points. Therefore, the collapse load in this case is an approximation to the lower bound of the collapse load.

3. DECOMPOSITION PROCEDURE

The optimum plastic structural design problem (1)–(4) can be treated as a nonlinear, nonconvex mathematical programming problem. Many nonlinear programming algorithms such as penalty functions, feasible directions and multiplier methods, etc. can be applied to solve this problem. However, most of these methods do not offer theoretical assurance that the solution obtained is a global optimum for the nonconvex problem. A solution procedure based on the Dantzig–Wolfe decomposition principle [19, 20] will be developed to solve the plastic design problem.

Without any loss of generality, the r.h.s. of eqn (2) is assumed to be nonnegative. Define

$$\sigma_j = \begin{Bmatrix} \sigma_{1j} \\ \sigma_{2j} \\ \vdots \\ \sigma_{Ij} \end{Bmatrix} \quad \mathbf{Q}_j(\sigma_j) = \begin{Bmatrix} \mathbf{Q}_j(\sigma_{1j}) \\ \mathbf{Q}_j(\sigma_{2j}) \\ \vdots \\ \mathbf{Q}_j(\sigma_{Ij}) \end{Bmatrix}, \quad \text{and } \mathbf{P} = \begin{Bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \vdots \\ \mathbf{P}_I \end{Bmatrix} \tag{7}$$

and

$$S_j = \{\sigma_j | \sigma_{ij} \in S_{ij};\ i = 1, \ldots, I\}. \tag{8}$$

The problem (1)–(4) can be restated as:

minimize
$$W = \sum_{j=1}^{J} c_j D_j \tag{9}$$

subject to
$$\sum_{j=1}^{J} \mathbf{Q}_j(\sigma_j) D_j = \mathbf{P} \tag{10}$$

$$\sigma_j \in S_j \tag{11}$$

$$D_j \ge 0 \tag{12}$$

$$j = 1, \ldots, J.$$

The set S_j is compact and convex on σ_j, since the sets S_{ij} are compact convex. Any point in a compact convex set can be expressed as the convex combination of the extreme points of this compact convex set. While the S_j associated with a truss member or a shear panel is a polyhedral convex set, the S_j for a CST element possesses an infinite number of extreme points. Let $\boldsymbol{\sigma}_j \in S_j$, then

$$\boldsymbol{\sigma}_j = \sum_n \lambda_{jn} \boldsymbol{\sigma}_{jn} \tag{13}$$

$$\sum_n \lambda_{jn} = 1 \tag{14}$$

$$\boldsymbol{\sigma}_{jn} \in E(S_j), \quad \lambda_{jn} \geq 0 \tag{15}$$

where $E(S_j)$ is the set of all the extreme points in set S_j. Since $\mathbf{Q}_j(\boldsymbol{\sigma}_j)$ is a linear function of $\boldsymbol{\sigma}_j$, it can be expressed as

$$\mathbf{Q}_j(\boldsymbol{\sigma}_j) = \sum_n \lambda_{jn} \mathbf{Q}_{jn}, \tag{16}$$

$$\mathbf{Q}_{jn} = \mathbf{Q}_j(\boldsymbol{\sigma}_{jn}). \tag{17}$$

Substitute (13)–(16) into (9)–(12) and let $D_{jn} = \lambda_{jn} D_j$, $D_{jn} \geq 0$, then the optimization problem becomes:

minimize
$$W = \sum_{j=1}^{J} \sum_n c_j D_{jn} \tag{18}$$

subject to
$$\sum_{j=1}^{J} \sum_n \mathbf{Q}_{jn}(\boldsymbol{\sigma}_{jn}) D_{jn} = \mathbf{P} \tag{19}$$

$$\boldsymbol{\sigma}_{jn} \epsilon E(S_j), \quad D_{jn} \geq 0, \quad j = 1, \ldots, J. \tag{20}$$

Problem (18)–(20) is equivalent to the original problem (9)–(12). It would be a linear programming problem if all the extreme points were known. The number of extreme points may be very large in fact, it is infinite for sets associated with CST elements. However, a particular point in S_j can be expressed as the convex combination of a finite number of extreme points in S_j. Therefore, only a finite number of extreme points in sets S_j are required to solve the problem (18)–(20). The difficulty is that the required extreme points are not known until a solution to the problem (18)–(20) is found. Fortunately, the Dantzig–Wolfe decomposition principle can be applied to generate useful extreme points.

Assuming that a finite number of extreme points, N, are known for each set S_j, the Nth restricted master problem can be written as:

minimize
$$W_N = \sum_{j=1}^{J} \sum_{n=1}^{N} c_j D_{jn} \tag{18A}$$

subject to
$$\sum_{j=1}^{J} \sum_{n=1}^{N} \mathbf{Q}_{jn} D_{jn} = \mathbf{P} \tag{19A}$$

$$D_{jn} \geq 0, \quad j = 1, \ldots, J, \quad n = 1, \ldots, N. \tag{20A}$$

Let $\boldsymbol{\pi}_N{}^t$ be a row vector of optimal simplex multipliers for the Nth restricted master,

$$\boldsymbol{\pi}_N{}^t = [\boldsymbol{\pi}_{1N}^t, \boldsymbol{\pi}_{2N}^t, \ldots, \boldsymbol{\pi}_{IN}^t] \tag{21}$$

where $\boldsymbol{\pi}_{iN}^t$ is a $1 \times M$ row vector associated with the equilibrium equations under the ith load condition. To determine which extreme points should be brought into the restricted master, the following subproblems have to be solved.

minimize $\quad c_j - \boldsymbol{\pi}_N{}^t \mathbf{Q}_j(\boldsymbol{\sigma}_j)$

s.t. $\quad \boldsymbol{\sigma}_j \in S_j$

$$j = 1, \ldots, J. \tag{22}$$

Each subproblem is associated with only one element. The optimal solution of the subproblem is an extreme point of S_j, since the objective function is linear on $\boldsymbol{\sigma}_j$ and S_j is a compact convex set defined by strictly convex yield functions. Let $\mathring{\boldsymbol{\sigma}}_j$ be the optimal solution for the subproblem, then, either of the following two cases may result;

$$c_j - \boldsymbol{\pi}_N{}^t\mathbf{Q}_j(\mathring{\boldsymbol{\sigma}}_j) < 0 \quad \text{for some } j.$$

Let $\boldsymbol{\sigma}_{j(N+1)} = \mathring{\boldsymbol{\sigma}}_j$, $\mathbf{Q}_{j(N+1)} = \mathbf{Q}_{j(N+1)}(\mathring{\boldsymbol{\sigma}}_j)$ and form the $(N+1)$st restricted master, since the additional columns introduced into the restricted master will reduce the value of the objective function W;

(ii) $$c_j - \boldsymbol{\pi}_N{}^t\mathbf{Q}_j(\mathring{\boldsymbol{\sigma}}_j) \geq 0 \text{ for all } j.$$

This means no additional columns can be introduced to reduce the value of the objective function. Therefore, the solution of the Nth restricted master is also a solution of the original problem.

It can be shown that the solution obtained by this procedure is a saddle point for the Lagrangian function associated with the problem (9)–(12), which is sufficient for global optimality. The optimal values for design variables and stresses are given by

$$\left.\begin{array}{l} \hat{D}_j = \sum_{n=1}^{N} \hat{D}_{jn} \\ \hat{\lambda}_{jn} = \hat{D}_{jn}/\hat{D}_j \\ \hat{\boldsymbol{\sigma}}_j = \sum_{n=1}^{N} \lambda_{jn}\boldsymbol{\sigma}_{jn} \end{array}\right\} \quad j = 1,\ldots,J, \quad n = 1,\ldots,N, \tag{23}$$

where $\hat{D}_{jn}$ is the solution for the Nth restricted master.†

Each subproblem (22) can be further decomposed into I second-level subproblems. Rewrite (22) as

$$\left.\begin{array}{ll} c_j + \min & \left(-\sum_{i=1}^{I} \boldsymbol{\pi}^t_{iN}\mathbf{Q}_j(\boldsymbol{\sigma}_{ij})\right) \\ \text{s.t.} & \boldsymbol{\sigma}_{ij} \in S_{ij} \end{array}\right\} \quad i = 1,\ldots,I, \quad j = 1,\ldots,J. \tag{24}$$

Since sets S_{ij} are independent of one another, the following second-level subproblems are obtained:

$$\left.\begin{array}{ll} \min & -\boldsymbol{\pi}^t_{iN}\mathbf{Q}_j(\boldsymbol{\sigma}_{ij}) \\ \text{s.t.} & \boldsymbol{\sigma}_{ij} \in S_{ij} \end{array}\right\} \quad i = 1,\ldots,I, \quad j = 1,\ldots,J. \tag{25}$$

Each second level subproblem is associated with a single element and a single load condition. The objective function is linear on $\boldsymbol{\sigma}_{ij}$ and there is only one constraint in each second-level subproblem. Using eqn (6) and the Kuhn–Tucker condition, the closed-form solution can be derived from the following equations:

$$\Phi_j(\mathring{\boldsymbol{\sigma}}_{ij}) = \bar{\sigma}_j^2 \tag{26}$$

$$\left\{\frac{\partial}{\partial \boldsymbol{\sigma}_{ij}}(\boldsymbol{\pi}^t_{iN}\mathbf{Q}_j(\boldsymbol{\sigma}_{ij}))\right\} = \gamma\left\{\frac{\partial \Phi_j}{\partial \boldsymbol{\sigma}_{ij}}\right\} \quad \text{at} \quad \boldsymbol{\sigma}_{ij} = \mathring{\boldsymbol{\sigma}}_{ij} \tag{27}$$

where $\gamma > 0$ is the Kuhn–Tucker Multiplier.

Since [18]

$$\mathbf{Q}_j(\boldsymbol{\sigma}_{ij}) = [B]_j\mathbf{F}_j(\boldsymbol{\sigma}_{ij}) = [B]_j[T]_j\boldsymbol{\sigma}_{ij}, \tag{28}$$

†In case of $\hat{D}_j = 0$, $\hat{\lambda}_{jn}$ may assume any value provided that eqns (14) and (15) are satisfied.

eqn (27) becomes:

$$[\pi_{iN}^t[B]_j[T]_j]^t = \gamma\left\{\frac{\partial\Phi_j}{\partial\sigma_{ij}}\right\} \quad \text{at} \quad \sigma_{ij} = \mathring{\sigma}_{ij}. \tag{29}$$

If the π_{iN}^t are viewed as nodal velocities under the ith load condition, the interpretation of eqn (29) becomes very interesting. Let

$$\dot{\mathbf{u}}_i = \pi_{iN}, \quad \bar{\gamma} = \frac{\rho_i}{c_j}\gamma \geq 0 \tag{30}$$

then

$$\begin{aligned}\frac{\rho_i}{c_j}\pi_{iN}^t[B]_j[T]_j &= \frac{\rho_i}{c_j}\dot{\mathbf{u}}_i^t[B]_j[T]_j\\ &= \dot{\epsilon}_{ij}^t\end{aligned} \tag{31}$$

which is the row vector of strain rates for the jth element under the ith load condition. The eqn (29) can be written as

$$\dot{\epsilon}_{ij} = \bar{\gamma}\left\{\frac{\partial\Phi_j}{\sigma_{ij}}\right\} \quad \text{at} \quad \sigma_{ij} = \mathring{\sigma}_{ij} \tag{32}$$

which is exactly the flow rule for rigid-perfectly plastic materials ([16], p. 14). The explicit expressions for the second-level subproblem solutions are given in Ref. [18]. The algorithm can be summarized as follows.

Do *Steps* 1–6 until satisfied.

Step 1. Let $N = 1$, $\pi_1{}' = [1, 1, \ldots, 1]$.

Step 2. Solve the subproblems (22) in terms of the second-level subproblems (25). Let $\mathring{\sigma}_j$ denote the optimal solution.

Step 3. If $c_j - \pi_N{}'Q_j(\mathring{\sigma}_j) \geq 0$ for all j, stop. The solution for the Nth restricted master is optimal. Otherwise, go to *Step* 4.

Step 4. Let $\sigma_{j(N+1)} = \mathring{\sigma}_j$, $Q_{j(N+1)} = Q_{j(N+1)}(\mathring{\sigma}_j)$, and form the $(N+1)$st restricted master, which is a linear programming problem.

Step 5. Solve the $(N+1)$st restricted master by simplex method and obtain an optimal simplex multipliers vector π_{N+1}^t.

Step 6. Let $N = N+1$ and go to *Step* 2.

The decomposition procedure breaks up the original problem into a restricted master and a number of subproblems, which are much smaller problems. Since the subproblems have simple closed-form solutions, it costs hardly anything to compute a numerical solution compared to that needed for the solution of the restricted master.

4. GLOBAL OPTIMALITY

The convergence theorem given in Ref. [20] is sufficient for the solution to be a global optimum. Nevertheless, the global optimality will be shown from a different point of view, i.e. the solution obtained by the algorithm developed in Section 3 is a saddle point for the Lagrangian function associated with the problem (9)–(12).

Let

$$\mathbf{D} = \begin{Bmatrix} D_1\\ D_2\\ \vdots\\ D_J\end{Bmatrix}, \quad \sigma = \begin{Bmatrix}\sigma_1\\ \sigma_2\\ \vdots\\ \sigma_J\end{Bmatrix}, \tag{33}$$

$$S = \{\sigma | \sigma_j \in S_j, j = 1, 2, \ldots, J\}. \tag{34}$$

The Lagrangian function can be written as

$$L(\boldsymbol{\pi};\mathbf{D},\boldsymbol{\sigma})=\sum_{j=1}^{J}c_jD_j-\boldsymbol{\pi}'\left(\sum_{j=1}^{J}\mathbf{Q}_j(\boldsymbol{\sigma}_j)D_j-\mathbf{P}\right)$$

$$=\boldsymbol{\pi}'\mathbf{P}+\sum_{j=1}^{J}(c_j-\boldsymbol{\pi}'\mathbf{Q}_j(\boldsymbol{\sigma}_j))D_j \tag{35}$$

$$\boldsymbol{\sigma}\in S \quad \text{and} \quad \mathbf{D}\geq 0. \tag{36}$$

A point $(\hat{\boldsymbol{\pi}};\hat{\mathbf{D}},\hat{\boldsymbol{\sigma}})$ is a saddle point for $L(\boldsymbol{\pi};\mathbf{D},\boldsymbol{\sigma})$ if and only if the following conditions are satisfied.

(i) $(\hat{\mathbf{D}},\hat{\boldsymbol{\sigma}})$ solves the problem $\min\limits_{\boldsymbol{\sigma}\in S,\,\mathbf{D}\geq 0} L(\hat{\boldsymbol{\pi}};\mathbf{D},\boldsymbol{\sigma})$;

(ii) $\sum\limits_{j=1}^{J}\mathbf{Q}_j(\hat{\boldsymbol{\sigma}}_j)\hat{D}_j-\mathbf{P}=0$;

(iii) $\hat{\boldsymbol{\pi}}'(\sum\limits_{j=1}^{J}\mathbf{Q}_j(\hat{\boldsymbol{\sigma}}_j)\hat{D}_j-\mathbf{P})=0$.

Note that condition (iii) is always satisfied if condition (ii) is satisfied.

Let the Nth restricted master be the final restricted master and $\hat{D}_{jn}$ be the solution of the final restricted master. Substituting (23) into (19A), yields:

$$\sum_{j=1}^{J}\sum_{n=1}^{N}\mathbf{Q}_j(\hat{\boldsymbol{\sigma}}_{jn})\hat{D}_{jn}=\sum_{j=1}^{J}\hat{D}_j\sum_{n=1}^{N}\mathbf{Q}_j(\hat{\boldsymbol{\sigma}}_{jn})\frac{\hat{D}_{jn}}{\hat{D}_j}$$

$$=\sum_{j=1}^{J}\mathbf{Q}_j(\hat{\boldsymbol{\sigma}}_j)\hat{D}_j=\mathbf{P}.$$

Therefore, condition (ii) is satisfied.

The solution for $\min L(\hat{\boldsymbol{\pi}};\mathbf{D},\boldsymbol{\sigma})$ exists if and only if

$$c_j-\hat{\boldsymbol{\pi}}'\mathbf{Q}_j(\boldsymbol{\sigma}_j)\geq 0,\quad \boldsymbol{\sigma}_j\in S_j \quad \text{for all } j. \tag{37}$$

The solution of $\min L(\hat{\boldsymbol{\pi}};\mathbf{D},\boldsymbol{\sigma})$ is given by

$$\min_{\boldsymbol{\sigma}\in S,\,\mathbf{D}\geq 0} L(\hat{\boldsymbol{\pi}};\mathbf{D},\boldsymbol{\sigma})$$

$$=\hat{\boldsymbol{\pi}}'\mathbf{P}+\min_{\boldsymbol{\sigma}\in S,\,\mathbf{D}\geq 0}\sum_{j=1}^{J}(c_j-\hat{\boldsymbol{\pi}}'\mathbf{Q}_j)D_j$$

$$=\hat{\boldsymbol{\pi}}'\mathbf{P}. \tag{38}$$

Since the optimization process terminates with:

$$\min_{\boldsymbol{\sigma}_j\in S_j} c_j-\hat{\boldsymbol{\pi}}'\mathbf{Q}_j(\hat{\boldsymbol{\sigma}}_j)\geq 0 \quad \text{for all } j,$$

the nonnegativity requirement (37) is satisfied at $\boldsymbol{\sigma}_j=\hat{\boldsymbol{\sigma}}_j$. Condition (i) will be satisfied if the following relation holds:

$$(c_j-\hat{\boldsymbol{\pi}}'\mathbf{Q}_j(\hat{\boldsymbol{\sigma}}_j))\hat{D}_j=0 \quad \text{for all } j. \tag{39}$$

The above equation is valid for any $\hat{D}_j=0$. If $\hat{D}_j>0$, there is at least one $\hat{D}_{jn}>0$. The only way $\hat{D}_{jn}$ can be greater than zero is that its associated column is in the optimal basis of the final restricted master. Therefore,

$$c_j-\hat{\boldsymbol{\pi}}'\mathbf{Q}_j(\hat{\boldsymbol{\sigma}}_{jn})=0 \quad \text{for} \quad n\in\{n|\hat{D}_{jn}>0\}.$$

Since $\hat{\lambda}_{jn} > 0$ for $\hat{D}_{jn} > 0$ and $\hat{\lambda}_{jn} = 0$ for $\hat{D}_{jn} = 0$, hence

$$\hat{\lambda}_{jn}(c_j - \hat{\boldsymbol{\pi}}'\mathbf{Q}_j(\hat{\boldsymbol{\sigma}}_{jn})) = 0 \quad \text{for all } n;$$

$$\sum_{n=1}^{N} \hat{\lambda}_{jn}(c_j - \hat{\boldsymbol{\pi}}'\mathbf{Q}_j(\hat{\boldsymbol{\sigma}}_{jn})) = 0;$$

$$c_j - \hat{\boldsymbol{\pi}}'\mathbf{Q}_j\left(\sum_{j=1}^{N} \hat{\lambda}_{jn}\hat{\boldsymbol{\sigma}}_{jn}\right) = 0;$$

$$c_j - \hat{\boldsymbol{\pi}}'\mathbf{Q}_j(\hat{\boldsymbol{\sigma}}_j) = 0 \quad \text{for} \quad \hat{D}_j > 0.$$

Therefore,

$$(c_j - \hat{\boldsymbol{\pi}}'\mathbf{Q}_j(\hat{\boldsymbol{\sigma}}_j))\hat{D}_j = 0 \quad \text{for} \quad \hat{D}_j > 0,$$

and condition (i) is satisfied. $(\hat{\boldsymbol{\pi}}; \hat{\mathbf{D}}, \hat{\boldsymbol{\sigma}})$ is a saddle point for $L(\boldsymbol{\pi}; \mathbf{D}, \boldsymbol{\sigma})$, which is sufficient for global optimality.

The number of nonzero design variables in the optimal solution for the plastic design of structures modeled by truss members and shear panels cannot exceed the number of equilibrium equations ($M \times I$). This means that an optimal plastic structure is statically determinate under a single load condition. Therefore, in the single-load case, an optimal plastic design solution is also a solution for the corresponding elastic design problem with stress limits, since the compatibility equations are not needed to calculate the stress distribution in a statically determinate structure.

5. DUALITY

The dual for the plastic design problem and its structural interpretation will be given in this section. Let (1)–(4), or its equivalent form (9)–(12), be the primal problem. A structure is considered to be feasible if the element sizes are all nonnegative and there exists a stress field which satisfies the equilibrium equations and the yield constraints. A feasible structure can usually be found for a well formulated structural optimization problem. The dual function and dual feasible region can be defined as:

$$f(\boldsymbol{\pi}) = \min_{\boldsymbol{\sigma} \in S, \mathbf{D} \geq 0} L(\boldsymbol{\pi}; \mathbf{D}, \boldsymbol{\sigma}) \tag{40}$$

and

$$U = \{\boldsymbol{\pi} | \min_{\boldsymbol{\sigma} \in S, \mathbf{D} > 0} L(\boldsymbol{\pi}; \mathbf{D}, \boldsymbol{\sigma}) \text{ exists}\} \tag{41}$$

where the Lagrangian function L is given by (35). The dual problem can be stated as

$$\begin{aligned} \max \quad & \mathbf{f}(\boldsymbol{\pi}), \\ \text{s.t.} \quad & \boldsymbol{\pi} \in U. \end{aligned} \tag{42}$$

The minimum of $L(\boldsymbol{\pi}; \mathbf{D}, \boldsymbol{\sigma})$ exists if and only if (37) is true, since otherwise the minimum is negative infinity. Thus,

$$U = \{\boldsymbol{\pi} | \boldsymbol{\pi}'\mathbf{Q}_j(\boldsymbol{\sigma}_j) \leq c_j \text{ for all } \boldsymbol{\sigma}_j \in S_j, \; j = 1, \ldots, J\}. \tag{43}$$

For any $\boldsymbol{\pi} \in U$, the minimum of $L(\boldsymbol{\pi}; \mathbf{D}, \boldsymbol{\sigma})$ is obtained by choosing $(c_j - \boldsymbol{\pi}'\mathbf{Q}_j(\boldsymbol{\sigma}_j))D_j = 0$, $j = 1, \ldots, J$, and then it follows that

$$\mathbf{f}(\boldsymbol{\pi}) = \boldsymbol{\pi}'\mathbf{P}. \tag{44}$$

Let $\pi^t = \dot{u}^t = [\dot{u}_1^t, \dot{u}_2^t, \ldots, \dot{u}_I^t]$ be a kinematically admissible velocity field, then

$$\pi^t Q_j(\sigma_j) = \frac{c_j}{\rho_j} \sum_{i=1}^{I} \dot{\epsilon}_{ij}^t \sigma_{ij}. \tag{45}$$

Substituting (45) into (43), the dual for the optimal plastic structural design problem can be written as follows:

max
$$\sum_{i=1}^{I} \dot{u}_i^t P_i \tag{46}$$

s.t.
$$\sum_{i=1}^{I} \dot{\epsilon}_{ij}^t \sigma_{ij} \le \rho_j \quad \text{for all} \quad \sigma_{ij} \in S_{ij}, \quad j = 1, \ldots, J. \tag{47}$$

The dual objective function is the total power of applied loads for a given kinematically admissible velocity field. The lefthand side of dual constraints represent the total specific power of dissipation in each element produced by the strain rates associated with the velocity field. The specific power of dissipation is the energy dissipation rate per unit length for truss members, or per unit area for shear panels and CST elements. A velocity field is feasible for the dual problem, if it is kinematically admissible and the associated total specific power of dissipation does not exceed the material density in each element.

By the lower bound theorem[21], the weight of any feasible structure cannot be smaller than the total power of the loads for any feasible velocity field. Therefore, a lower bound for the minimum weight can be obtained from any feasible velocity field. The optimal simplex multipliers $\hat{\pi}$ and the solution $(\hat{D}, \hat{\sigma})$ were shown in the previous section to be a saddle point for the Lagrangian function associated with the primal. The saddle point theorem states that:

$$\sum_{j=1}^{J} c_j \hat{D}_j = \sum_{i=1}^{I} \hat{\pi}_i^t P_i \tag{48}$$

and $\hat{\pi}$ is a solution for the dual. Therefore, the optimal simplex multipliers $\hat{\pi}$ can be treated as the velocity field or collapse mechanism associated with an optimal design.

6. IMPLEMENTATION OF DECOMPOSITION PROCEDURE

In this section some details for implementation of the decomposition procedure, developed in Section 3, are described. It was shown in Ref. [18] that the body force, design variable linking and side constraints, can be easily incorporated into the decomposition procedure.

Primary modules of the computer program which implement the decomposition procedure are shown in Fig. 1. The structural model generator formulates the plastic design problem for two- and three-dimensional structures which can be represented by truss members, rectangular shear panels and constant stress triangular membrane elements.

All elements with the same orientation and configuration are grouped together and a single unit element force vector F_j is computed for each group. However, the boolean assembly matrix $[B]_j$ must be generated for every element, because each element is usually connected to a different set of nodes. Due to the requirement for a positive r.h.s. in the simplex method, the negative components in the load vector are changed to positive and a record is kept in a sign indication vector.

The information contained in the structural model may not be in the best form for application of the decomposition algorithm. Further manipulations are necessary to improve numerical stability and the speed of convergence. The primary concern is the linear programming model of the restricted master, since the subproblems can be easily solved. Because scaling can improve conditioning and reduce the range of variables[22], it is performed by the mathematical programming model generator. Using the scaled model the optimal simplex multipliers generated by the decomposition procedure also involve a scaling factor. The details of the scaling procedure employed are given in Ref. [18].

The restricted master problem is solved by the revised simplex method with product form

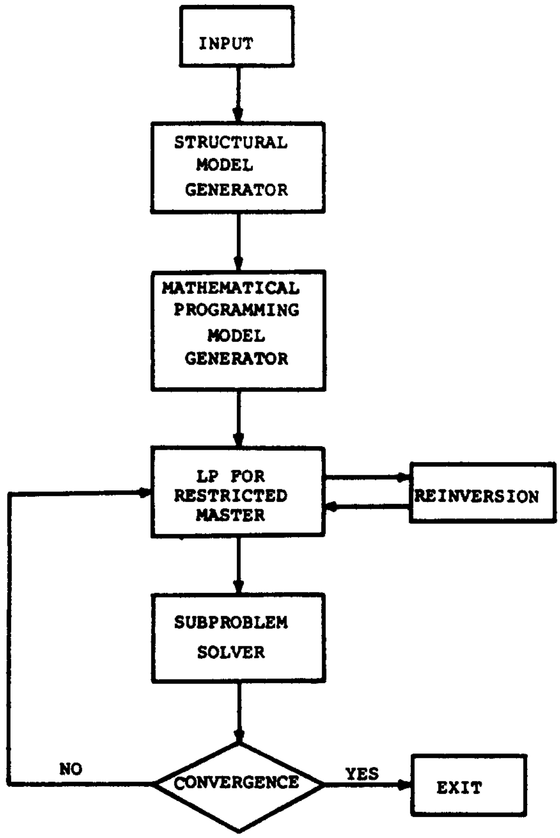


Fig. 1. Flow chart for decomposition procedure.

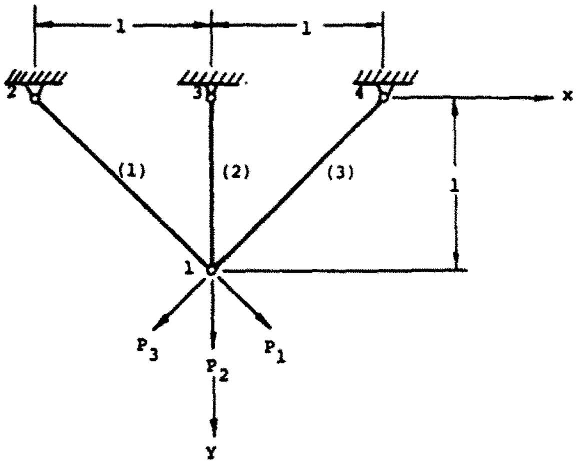
inverse of the basis matrix. Periodic reinversions are used to truncate the list of elementary matrices and improve numerical stability. The details of the 2-phase simplex method can be found in Refs. [20, 21]. The decomposition procedure starts with a simplex multiplier vector $\pi = [1, 1, \ldots, 1]$ which can be regarded as a phase I simplex multiplier vector for the first restricted master problem, before any pivot operation is executed. The phase I restricted master will determine the feasibility of the original problem and find an initial feasible solution. An optimal solution and the associated optimal simplex multiplier vector are obtained in phase II. It should be noted that the c_j are set at zero in phase I. A good inversion routine plays a central role in the success of a linear programming system. The purpose of reinversion is to reduce the number of elementary matrices representing the inverse of the basis matrix and to maintain the spareness of the eta vectors (see Appendix B of Ref. [18]). A reinversion routine used here is based on Larson's procedure [21], modified by Tomlin's pivot selection criteria [23]. The frequency of reinversion in the solution process depends on the density and the number of the generated eta vectors.

7. NUMERICAL EXAMPLES

The decomposition algorithm for the plastic structural design optimization problems was coded in FORTRAN. The solutions for examples reported in this section were obtained on a CDC-6600 computer without using any auxiliary storage. One iteration in the decomposition algorithm is defined as the computations required for the solution of a restricted master problem and the associated subproblems.

3-*Bar truss*

The first example problem is a 3-bar planar truss (see Fig. 2) for which results have been previously reported for the plastic design (labeled as first LBM) in Ref. [24]. The truss is subject to three independent load conditions. The difference between yield stresses is quite pronounced, as shown in Fig. 2. The minimum weight and optimal member sizes listed in Table



Material properties: E = 1 ρ = 1

Yield stresses: ±5 for members 1 and 3

±20 for member 2

Load conditions: 1. P_1 = 40

2. P_2 = 30

3. P_3 = 30

Fig. 2. 3-Bar truss.

1 are essentially the same as those reported in Ref. [24]. The velocity fields shown in Table 2 are obtained from the optimal simplex multipliers in the final restricted master. The velocities and strain rates contain a scaling factor of $\sigma_{max}/(l_j)_{max} = 10\sqrt{2}$. It is interesting to note that the velocities are zero under load condition 3. This is due to the fact that members 1 and 2 are not stressed to their limits, as shown in Table 3, consequently, the truss does not collapse under load condition 3. According to the flow theory of plasticity, the strain rate is zero for any member not stressed to its yield limit, even in a collapse mode. Member 1 in load condition 2 exhibits this characteristic, since its strain rate is zero and the stress is below the yield limit, as shown in Table 3. The collapse mechanism under load condition 2 is very interesting. Although the applied load P_2 points downward, the resultant of velocities at node 1 lies in the direction

Table 1. Optimal design for 3-bar truss

Member size			Weight
1	2	3	
5.8786	0.7500	2.1213	12.0636

Table 2. Velocity field for 3-bar truss

Node		Load condition 1	2	3
1	X	5.3033	−0.3536	0
	Y	0.3536	0.3536	0

Table 3. Stresses and strain rates for 3-bar truss

Member	Load condition 1 Stress	Load condition 1 Strain Rate	Load condition 2 Stress	Load condition 2 Strain Rate	Load condition 3 Stress	Load condition 3 Strain Rate
1	5.0	2.8284	1.8	0	−1.6	0
2	20.0	0.3536	20.0	0.3536	17.7	0
3	−5.0	−2.4749	5.0	0.3536	5.0	0

perpendicular to member 1. This can be interpreted in terms of the stresses and strain rates for load condition 2. Members 2 and 3 are stressed to their yield limits, but member 1 does not flow. Therefore, the rotation of member 1 about node 2 is the only physically possible motion during collapse and this produces a linear velocity perpendicular to member 1 at node 1.

Square panel

The first example involving constant stress triangular membrane elements is a square panel (see Fig. 3) for which optimal design results have been previously reported in Ref. [12]. A uniformly distributed inplane side load of 500 kg/cm is applied to one edge of the panel as shown in Fig. 3. The panel is assumed to be in a state of plane stress.

Finite element models were constructed by first dividing the panel into squares, and then each square was divided into two triangles. A total of four models were used to generate optimal plastic designs. The largest model, which consists of 72 elements is shown in Fig. 3. The number of equilibrium equations, which is equal to the number of rows in the LP program for the restricted master, ranged from 24 for the 18-element model to 84 for the 72-element model. The optimal plastic design problems were solved for each of these models with linking imposed in such a way that the two CST elements within each square are required to have the same thickness. Minimum volume vs finite element mesh refinement is shown in Fig. 4. It can be seen that the minimum volume increases as the finite element mesh becomes finer. This phenomenon can be attributed to the approximate nature of the finite element analysis, since CST elements tend to underestimate the stresses in a coarse model. The optimal thickness (cm) distribution for the 72-element model is shown in Fig. 5. The elements near the lower right-hand corner have small thicknesses. All elements in each model were stressed to their yield limits at the optimal solutions. The phase *II* iteration histories for the 72-element models are given in Fig. 6. Fifteen phase *I* and *II* iterations were required to converge the solution for each model.

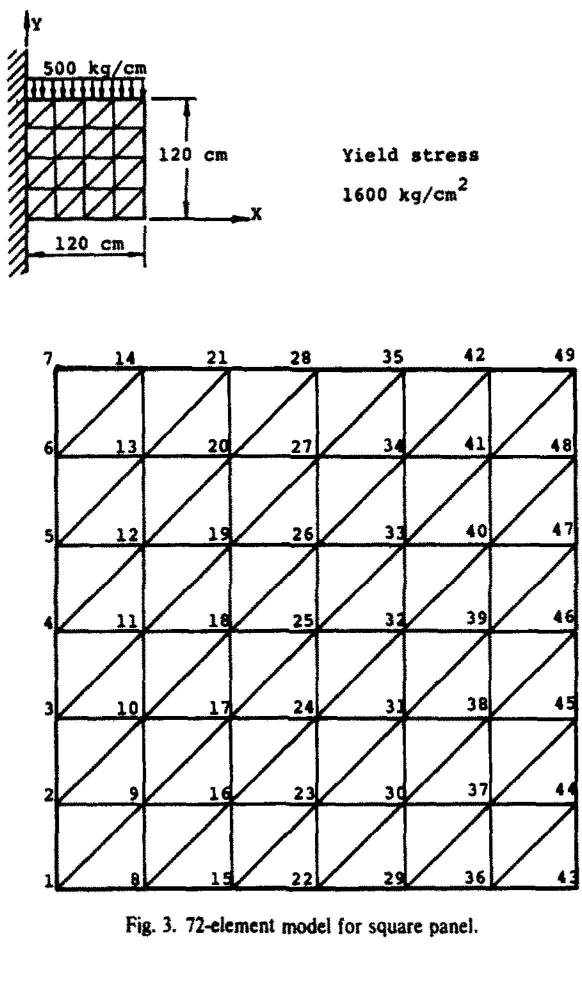


Fig. 3. 72-element model for square panel.

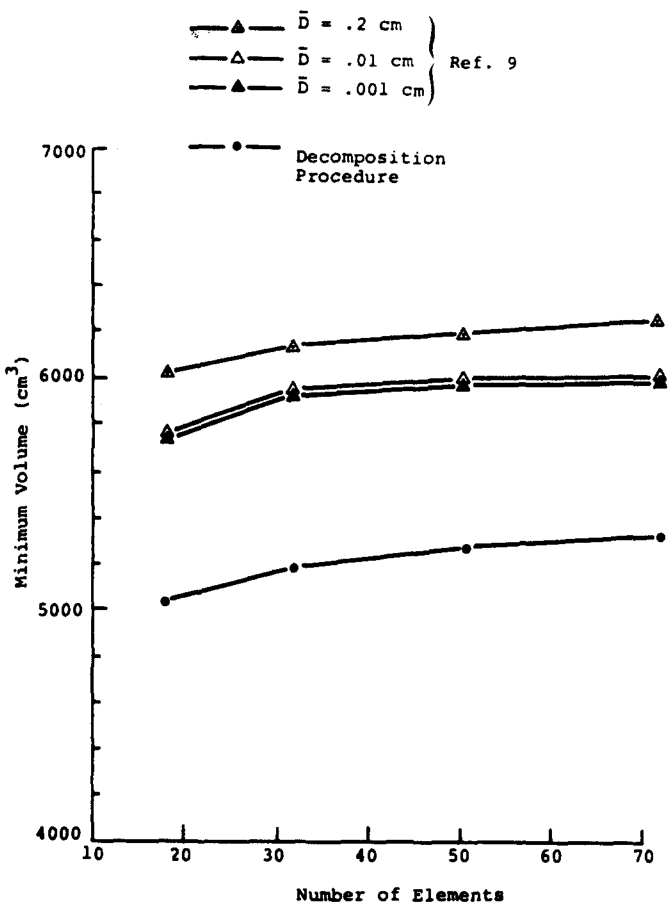


Fig. 4. Comparison of optimal designs for square panel.

2.132	.7802	.5257	.3958	.3847	.2742
0	.8288	.5122	.4228	.4170	.1795
.3742	.3139	.4233	.5092	.3307	.0956
.2345	.2452	.5793	.4243	.1601	.0270
.2213	.7218	.3963	.1470	.0312	.0024
.8947	.2654	.0719	.0164	.0018	.00038

Note: Optimal thicknesses are given in centimeters

Fig. 5. Optimal design for square panel 72 element model

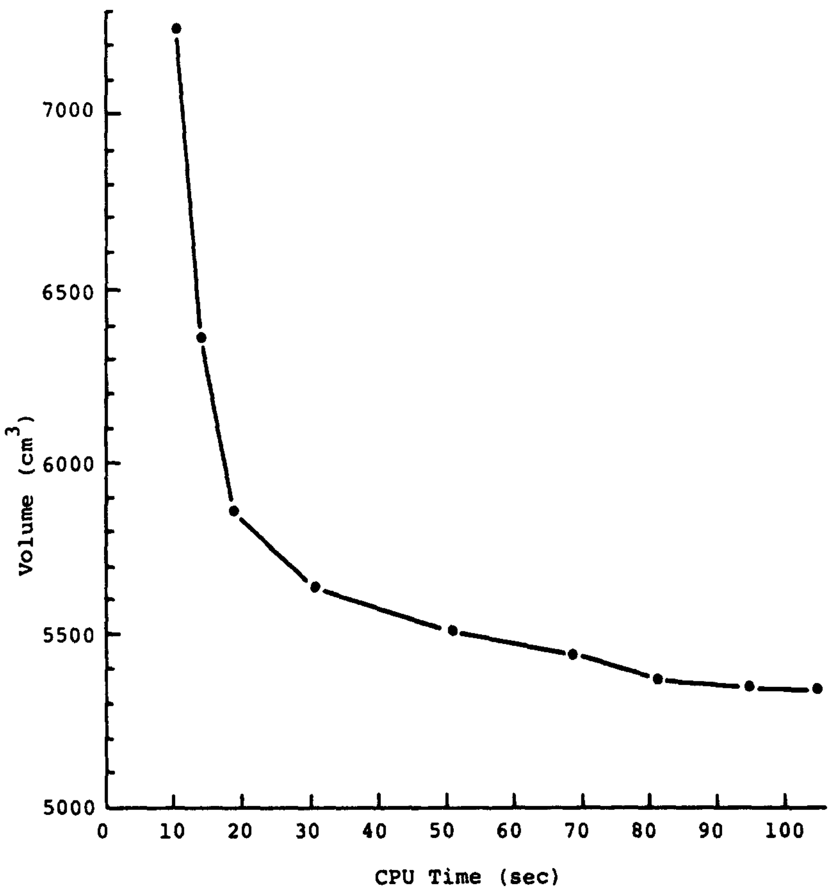


Fig. 6. Phase *II* iteration history for square panel, 72-element model.

The same problem was treated in Ref. [12]. The approach reported in Ref. [12] consisted of CST element discretization of plane stress structures, piecewise linearization of yield surfaces and solving a large LP problem with element thicknesses and forces as the variables. Their finite element models for the square panel are exactly the same as those used in the decomposition procedure. However, Von Mises yield criteria were linearized by seven pairs of parallel planes for each element and the number of constraints in the resulting LP problem ranged from 276 for the 18-element model to 1080 for the 72-element model. These LP problems are much larger than those for the restricted master in the decomposition procedure.

The minimum volumes reported in Ref. [12] are plotted in Fig. 4 for three different minimum thickness requirements. A comparison between the solutions generated by the decomposition procedure and those reported in Ref. [12] indicates that the minimum volumes reported in Ref. [12] are 14% higher than those obtained by the decomposition procedure. The large volume penalty is not surprising, since the approach used in Ref. [12] can generate only an approximate solution because it replaces the Von Mises yield criteria with a conservative approximation (i.e. a set of linear facets). The CPU time reported in Ref. [12], using a UNIVAC 1108 computer, ranged from 49 sec for an 18-element model to 3120 sec for a 98-element model. Figure 7 shows the CPU time required to obtain solutions on a CDC-6600 computer by the decomposition procedure.

Box beam

The last example involves the optimal plastic design of a cantilever box beam, shown in Fig. 8. The beam consists of two flat cover sheets parallel to the X,Y-plane, four spars in the X-direction and four ribs in the Y-direction. The yield stresses (40 ksi) and material densities (0.1 pci) were uniform for all structural members. A set of loads which will induce both bending and torsion was applied at the free end. Based on symmetry considerations a finite element model representing the upper half of the box beam was used in the decomposition procedure.

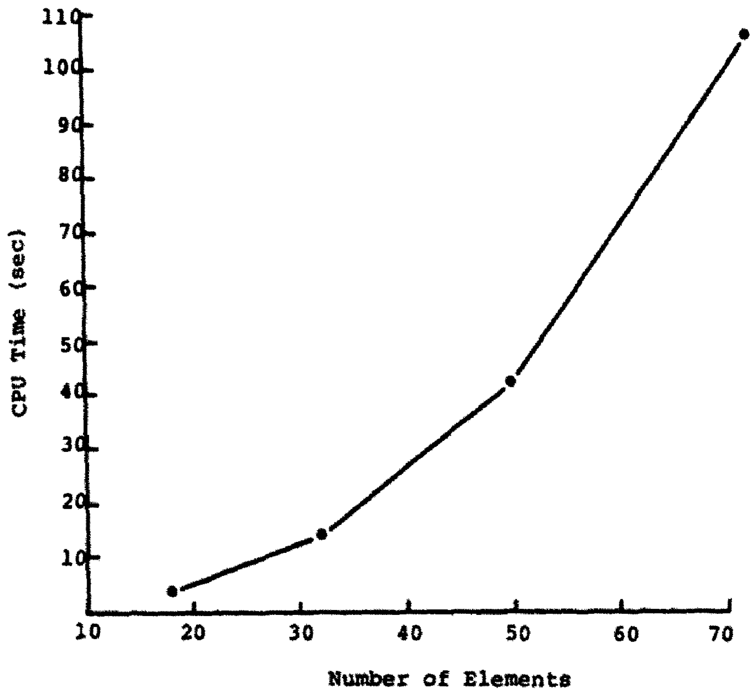


Fig. 7. CPU time for square-panel.

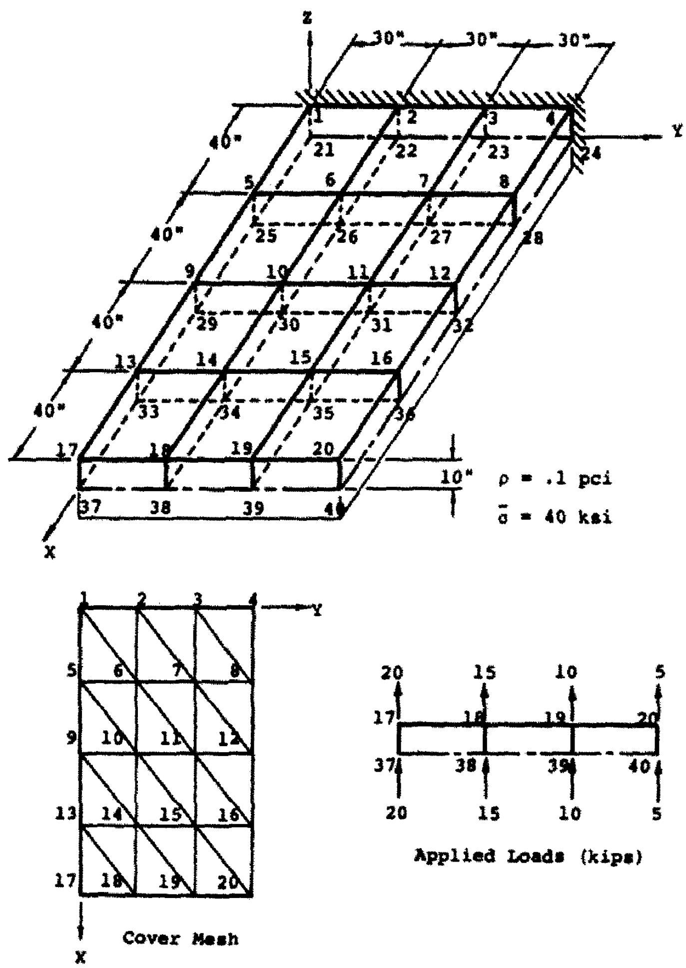


Fig. 8. Finite element model for box beam.

The numbers and the types of elements used in the model are summarized as follows: 24 CST elements for the cover sheet; 16 shear panels for the spar webs; 16 truss members for the spar caps; 12 shear panels for the rib webs; 12 truss members for the rib caps; and 16 truss members for posts at the intersections of the spars and ribs. The analysis degrees of freedom in the X and Y directions were suppressed at nodes 25–40 so that the deflections would satisfy antisymmetry requirements.

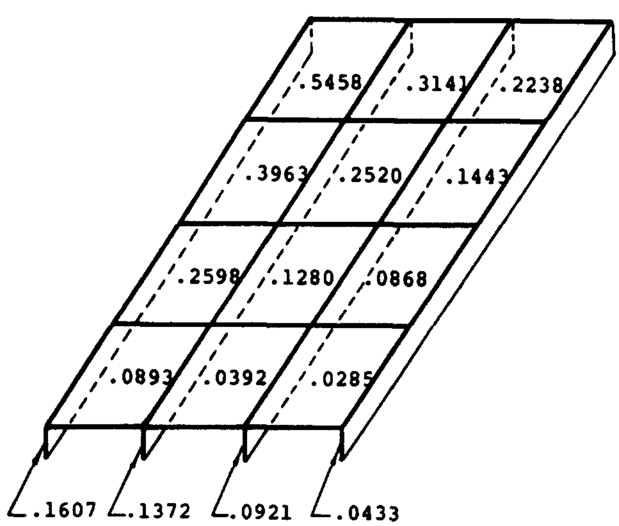
The element thicknesses at an optimal plastic design are displayed in Fig. 9. All the ribs (including caps and webs) and all the spar caps have vanished. The primary functions of the ribs are: (1) to transfer loads to the spars and (2) to prevent the cover sheets from buckling. In this example the loads were applied at nodal points where the spars are located and buckling of the cover sheets was not considered. Therefore, the optimization algorithm removed these unnecessary ribs to reduce the weight. It is more difficult to explain why the spar caps have vanished. Nevertheless, CST elements tend to underestimate the stresses in a coarse finite element model, and this appears to give the cover sheets an artificial advantage over the neighboring spar caps. The minimum weight and iteration history are shown in Fig. 10. It took 14.5 sec to obtain an optimal solution.

8. CONCLUDING REMARKS

An optimization procedure based on the Dantzig–Wolfe decomposition principle has been developed and successfully implemented for optimal plastic design of a rather general class of structures which can be represented by truss members, shear panels and CST elements. The decomposed structurel synthesis problems consist of a restricted master and a number of subproblems. These subproblems can be further divided into the second-level subproblems, for which closed form solutions have been obtained. Improvements in numerical efficiency are obvious, since the original problem is replaced by a number of smaller problems. In fact, this has been confirmed by the numerical examples and comparison with previously reported results.

The solution generated by the decomposition algorithm has been shown to be a saddle point for the associated Lagrangian function, which is sufficient for global optimality. The closed form solutions for the second level subproblems have been shown to comply with the flow rule in the theory of plasticity. Optimal simplex multipliers associated with the restricted master have been identified with velocity fields or collapse mechanisms at an optimal design. The physical significance of the relations between stresses, strain rates and collapse mechanisms has been brought out through a numerical example.

The dual problem has been formulated for the optimal plastic structural design. It is a problem of searching for kinematically admissible velocity fields such that the total power of



Note: Optimal thicknesses are given in inches.

Fig. 9. Optimal design for box beam.

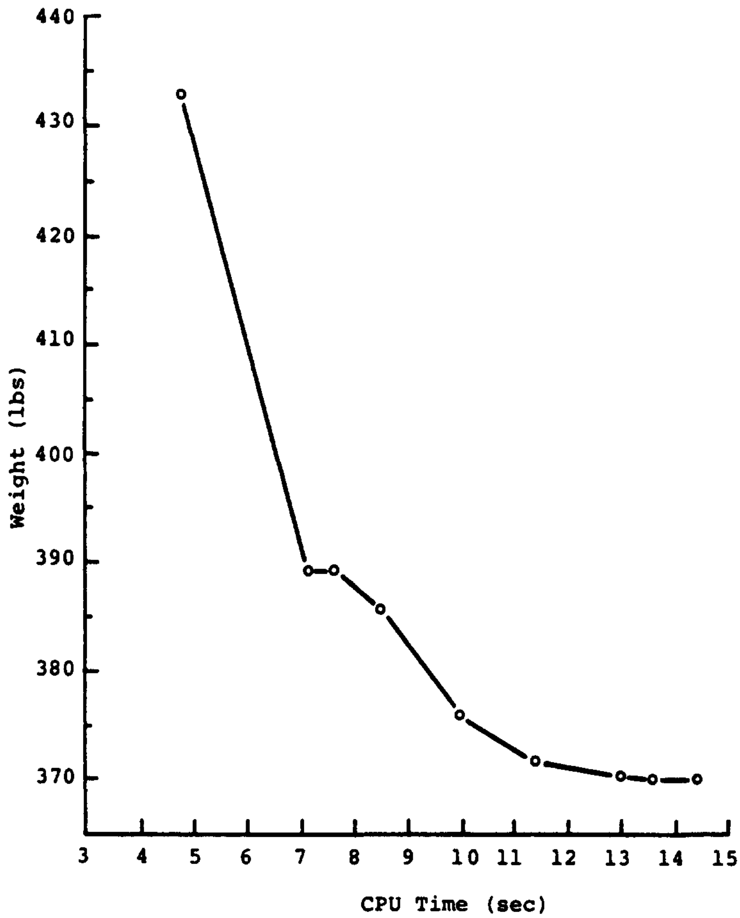


Fig. 10. Iteration history for box beam.

loads is maximized, subject to constraints requiring that the total specific power of dissipation in each element is less than or equal to its material density. Due to the existence of a saddle point, there is no duality gap, i.e. the minimum structural weight is equal to the maximum power of loads.

The decomposition procedure developed for the optimal plastic structural synthesis is efficient and possesses theoretical convergence. The bulk of the computational effort is involved in obtaining solutions for the restricted masters, since closed form solutions for subproblems can be easily calculated. The restricted master is a linear programming problem. Therefore, further efficiency gains can be achieved by:

(1) applying the Dantzig-Wolfe decomposition principle to the restricted master problem;

(2) reducing the number of columns in the restricted master via the column dropping procedures; and

(3) using a feasible starting point which can be generated by an elastic structural analysis to eliminate phase *I* iterations.

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